Recovering the M-channel Sturm-Liouville operator from M+1 spectra.

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Abstract

For a system of M coupled Schrödinger equations, the relationship is found between the vector-valued norming constants and $M+1$ spectra corresponding to the same potential matrix but different boundary conditions. Under a special choice of particular boundary conditions, this equation for norming vectors has a unique solution. The double set of norming vectors and associated spectrum of one of the $M+1$ boundary value problems uniquely specifies the matrix of potentials in the multichannel Schrödinger equation.

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1 Introduction

Consider the system of coupled one-dimensional Schrödinger equations

$$-\frac{d^2}{dx^2}\Psi_\alpha(x)+\sum_\beta V_{\alpha\beta}(x)\Psi_\beta(x)=(E-\varepsilon_\alpha)\Psi_\alpha(x),\ \ \alpha=1,...,M. \tag{1}$$

In this system, each equation is referred to as a 'channel' and ε_α's are the energies of channel 'thresholds'. Once $E\geq\varepsilon_\alpha$, it is said that α's threshold becomes open. The system (1) is a matrix generalization of the ordinary one-dimensional Schrödinger equation. The coupled Schrödinger equations originate in the Feshbach's unified theory [1] of nuclear reactions and correspond to so-called approximation of the strong coupling (when a finite number of equations in (1) is left). Now, that method, renewed and generalized (see, e.g. [2]), finds a lot of applications and, rightfully, is one of the most universal tools for microscopic description of systems with many degrees of freedom (nuclear structure, reactions, molecules, etc).

The inverse problem for multichannel Schrödinger equation (1) has also been developed [3-5]. As in one-channel case, one can uniquely restore the potential matrix $V_{\alpha\beta}(x)$

from the spectral measure that, e.g., for the case of bounded interval, is specified by the complete set of eigenvalues E_n and so-called norming vectors (spectral weight vectors) $\gamma_\alpha(E_n)$. These vectors characterize the behaviour of the normalized wave functions $\Psi_\alpha(x, E_n)$ at one of the boundaries of interval (or at the origin for a half-axis problem, etc.), see also below.

At the same time, in the one-channel case we have more variants of the inverse problem. Among them, there is a statement of inverse eigenvalue problem on a bounded interval where no norming constants occur. Namely, the potential is uniquely recovered from a knowledge of only two different spectra, each for a distinct pair of homogeneous boundary conditions (with the same potential) [6] . There were established necessary and sufficient conditions of the solvability of the inverse Sturm-Liouville problem from two-spectra, see, e.g., the book [7].

Till now, one attempt to generalize this theorem to the multichannel Sturm-Liouville operator has been known to the author – the article [8] (the case of a finite-difference operator). Though not complete, this work gave an idea of the existence of such a generalization in principle. No doubt, the possibility of deriving potential matrix from a certain set of spectra would contribute to the multichannel inverse problem theory. In present article, results concerning that problem are obtained. It is found that $M+1$ spectra determine the $V_{\alpha\beta}(x)$ and, under special conditions, it is possible to uniquely restore multichannel Sturm-Liouville operator.

The central idea of the paper is to derive the relationship between $M+1$ spectra and M-component norming vector $\gamma_\alpha(E_n)$ associated with one of the $M+1$ boundary value problems. Then, having the double set of eigenvalues and norming vectors, one can uniquely restore an interaction matrix (by Gel'fand-Levitan procedure).

Next section is devoted to setting forth these results. We shall find the sought expression which, however, does not guarantee the uniqueness in itself. Only under a special choice of boundary conditions it can be represented in a form of system of linear algebraic equations which give a simple criterion of the uniqueness and solvability. For the sake of the reader's convenience, the narrative is organized so that it goes partially in parallel with standard derivation of two spectra formulas given in [7], chapter 3.

2 Derivation of the formula for norming vector

We are beginning this section with preliminary notations. Let us rewrite the system (1) in a more symbolic form as follows

$$-\frac{d^2}{dx^2}y(x) + \hat{V}(x)y(x) = \lambda y(x), \quad x \in [0, a] \tag{2}$$

where y stands for the whole vector-column solution

$$y(x) \equiv \begin{pmatrix} \Psi_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_M(x) \end{pmatrix},$$

and

$$\hat{V} \equiv V_{\alpha\beta} + \varepsilon_\alpha, \quad \lambda \equiv E.$$

The potential matrix is the real symmetric matrix of continuous functions, $x \in [0, a]$. Next, we add to the equation (2) the following boundary conditions

$$\begin{cases} y'(0) - \hat{h}y(0) = 0 & y'(a) + \hat{H}y(a) = 0 \\ y_i'(0) - \hat{h}_i y_i(0) = 0 & y_i'(a) + \hat{H}y_i(a) = 0, \quad i = 1, ..., M \end{cases} \tag{3}$$

where we take $\hat{h}$, $\hat{h}_i$ and $\hat{H}$ to all be the real symmetric matrices. We denote the spectra of the M+1 problems (2) and (3) by $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda_n^i\}_{n=1}^\infty$, respectively. There is no theorem of interlacing of the spectra in M-channel case, $M > 1$. So, we additionally require that no spectrum degeneracy should occur.

Let us denote by $\hat{\phi}(x, \lambda)$ and $\hat{\chi}_i(x, \lambda)$ the matrix solutions of the equation (2) satisfying the initial conditions

$$\hat{\phi}(0, \lambda) = \hat{1}, \;\; \hat{\phi}'(x, \lambda)|_{x=0} = \hat{h}, \;\; \hat{\chi}_i(0, \lambda) = \hat{1}, \;\; \hat{\chi}_i'(x, \lambda)|_{x=0} = \hat{h}_i, \tag{4}$$

where the prime stands for the derivative with respect to x. In what follows we shall use the prime to denote this derivative apart from the special cases the reader will be let know of. Besides, the hat will always stand for the matrix. Do not confuse the following: a matrix solution of (2) means that each column of the matrix is a vector-solution, only satisfying a specific initial (boundary) condition. Eigenvalues of the boundary value problems (2) and (3) coincide with zeros of determinants of the matrices

$$\begin{cases} \hat{\Phi}(\lambda) = \bar{\hat{\phi}}'(x, \lambda)|_{x=a} + \bar{\hat{\phi}}(a, \lambda)\hat{H} \\ \hat{\Phi}_i(\lambda) = \bar{\hat{\chi}}_i'(x, \lambda)|_{x=a} + \bar{\hat{\chi}}_i(a, \lambda)\hat{H}, \end{cases} \tag{5}$$

where the bar sign denotes transpose.

Now we introduce the norming vectors associated with the spectrum $\{\lambda_n\}_{n=1}^\infty$

$$\gamma_{\lambda_n} \equiv \begin{pmatrix} \gamma_1(\lambda_n) \\ \cdot \\ \cdot \\ \cdot \\ \gamma_M(\lambda_n) \end{pmatrix},$$

so that

$$\hat{\phi}(x,\lambda_n)\gamma_{\lambda_n} = y(x,\lambda_n) \tag{6}$$

with $y'(x,\lambda_n)|_{x=a} + \hat{H}y(a,\lambda_n) = 0$ and $\int_0^a \sum_{\alpha=1}^M [\Psi_\alpha(x,\lambda_n)]^2 dx = 1$. Likewise, for the spectra $\{\lambda_n^i\}_{n=1}^\infty$

$$\gamma_{\lambda_n^i} \equiv \begin{pmatrix} \gamma_1(\lambda_n^i) \\ \cdot \\ \cdot \\ \cdot \\ \gamma_M(\lambda_n^i) \end{pmatrix},$$

so that

$$\hat{\chi}_i(x,\lambda_n^i)\gamma_{\lambda_n^i} = y_i(x,\lambda_n^i) \tag{7}$$

with $y_i'(x,\lambda_n^i)|_{x=a} + \hat{H}y_i(a,\lambda_n^i) = 0$ and $\int_0^a \sum_{\alpha=1}^M [\Psi_\alpha(x,\lambda_n^i)]^2 dx = 1$. Let us also introduce the function γ_λ (vs. λ) so that $\gamma_\lambda = \gamma_{\lambda_n}$ when $\lambda = \lambda_n$ and $\gamma_\lambda = \gamma_{\lambda_n^i}$ when $\lambda = \lambda_n^i$. That function makes the sense at the points λ_n and λ_n^i only. In between, we have the freedom to specify it arbitrarily. We can only require this function to be continuous differentiable and have no singularities.

We take

$$f_i(x,\lambda) \equiv \bar{\gamma}_\lambda \bar{\hat{\chi}}_i(x,\lambda) + m_i(\lambda)\bar{\gamma}_\lambda \bar{\hat{\phi}}(x,\lambda), \tag{8}$$

where $m_i(\lambda)$ is scalar and we require that

$$f_i'(x,\lambda)|_{x=a} + f_i(a,\lambda)\hat{H} = 0 \implies \tag{9}$$

$$m_i(\lambda)[\bar{\gamma}_\lambda \bar{\hat{\phi}}'(x,\lambda)|_{x=a} + \bar{\gamma}_\lambda \bar{\hat{\phi}}(a,\lambda)\hat{H}] = -[\bar{\gamma}_\lambda \bar{\hat{\chi}}_i'(x,\lambda)|_{x=a} + \bar{\gamma}_\lambda \bar{\hat{\chi}}_i(a,\lambda)\hat{H}]. \tag{10}$$

Comparing with (5) we have

$$m_i(\lambda) = -\frac{\bar{\Phi}_i(\lambda)\Phi(\lambda)}{\bar{\Phi}(\lambda)\Phi(\lambda)}, \tag{11}$$

where we denote $\Phi(\lambda) \equiv \bar{\hat{\Phi}}(\lambda)\gamma_\lambda$ and $\Phi_i(\lambda) \equiv \bar{\hat{\Phi}}_i(\lambda)\gamma_\lambda$.

Next, employing the well known Green formula we have

$$\begin{aligned}(\lambda-\lambda_n)\int_0^a f_i(x,\lambda)\hat{\phi}(x,\lambda_n)\gamma_{\lambda_n}dx = (\lambda-\lambda_n)\int_0^a \bar{\gamma}_\lambda \bar{\hat{\chi}}_i(x,\lambda)\hat{\phi}(x,\lambda_n)\gamma_{\lambda_n}dx \\ -(\lambda-\lambda_n)\frac{\bar{\Phi}_i(\lambda)\Phi(\lambda)}{\bar{\Phi}(\lambda)\Phi(\lambda)}\bar{\gamma}_\lambda \bar{\hat{\phi}}(x,\lambda)\hat{\phi}(x,\lambda_n)\gamma_{\lambda_n}dx = f_i'(x,\lambda)|_{x=0}\hat{\phi}(0,\lambda_n)\gamma_{\lambda_n} \\ -f_i(0,\lambda)\hat{\phi}(x,\lambda_n)|_{x=0}\gamma_{\lambda_n} = \bar{\gamma}_\lambda(\hat{h}_i - \hat{h})\gamma_{\lambda_n},\end{aligned} \tag{12}$$

where we use the definitions (8), (4) and (11). The last equality follows from the fact that the matrices in (3) are symmetric.

Let us pass to the limit $\lambda \to \lambda_n$. Then the equation (12) goes over into

$$-\frac{\frac{d}{d\lambda}[\bar{\Phi}(\lambda)\Phi(\lambda)]|_{\lambda=\lambda_n}}{\bar{\Phi}_i(\lambda_n)\Phi(\lambda_n)}\bar{\gamma}_{\lambda_n}(\hat{h}_i - \hat{h})\gamma_{\lambda_n} = 1, \tag{13}$$

where we used the L'Hospital rule.

We shall prove that this formula can be represented as

$$(\lambda_n^i - \lambda_n)^{-1}\prod_{\mu=1}^{\infty}{}'\frac{\lambda_\mu - \lambda_n}{\lambda_\mu^i - \lambda_n}\bar{\gamma}_{\lambda_n}(\hat{h}_i - \hat{h})\gamma_{\lambda_n} = 1, \tag{14}$$

where the prime denotes that we omitted, in the product, the term with the number n.

Since $\Phi(\lambda)$ and $\Phi_i(\lambda)$ are the entire holomorphic functions they are determined (to within constant multipliers) by their zeros and, hence, can be represented as follows

$$\Phi(\lambda) = C\prod_{\mu=1}^{\infty}(1 - \frac{\lambda}{\lambda_\mu}); \;\; \Phi_i(\lambda) = C_i\prod_{\nu=1}^{\infty}(1 - \frac{\lambda}{\lambda_\nu^i}). \tag{15}$$

Substituting (15) into (13) we have

$$\frac{\frac{1}{\lambda_n}\prod_{\mu=1}^{\infty}{}'(1 - \frac{\lambda_n}{\lambda_\mu})\bar{C}C}{\prod_{\nu=1}^{\infty}(1 - \frac{\lambda_n}{\lambda_\nu^i})\bar{C}_iC}\bar{\gamma}_{\lambda_n}(\hat{h}_i - \hat{h})\gamma_{\lambda_n} = 1. \tag{16}$$

Now we have to ascertain the expression for the $\bar{C}C/\bar{C}_iC$. We shall need some knowledge about an asymptotic behaviour of the solutions of (2). First of all, these equations become uncoupled in the limit $\lambda \to \infty$. So, as in one-channel case, we have $\lim_{\lambda\to\infty}\hat{\Phi}(\lambda)\{\hat{\Phi}_i(\lambda)\}^{-1} = 1$, and the same for the transpose of these matrices. Taking this into account we obtain

$$\frac{\bar{C}C}{\bar{C}_iC}\prod_{\mu=1}^{\infty}\frac{\lambda_\mu^i}{\lambda_\mu}\lim_{\lambda\to\infty}\prod_{\mu=1}^{\infty}\frac{\lambda_\mu - \lambda}{\lambda_\mu^i - \lambda} = 1. \tag{17}$$

We have the following asymptotic formulas for λ and λ^i: $\lambda_\mu = \mu^2 + O(1)$ and the same for λ^i. Then $\lambda_\mu^i - \lambda_\mu = O(1)$ and the series $\sum_\mu^\infty |(\lambda_\mu - \lambda_\mu^i)/(\lambda_\mu^i - \lambda)|$ converges uniformly as $\lambda \to \infty$. Hence, we can pass to the limit in each term of the infinite product

$$\lim_{\lambda\to\infty}\prod_{\mu=1}^{\infty}\frac{\lambda_\mu - \lambda}{\lambda_\mu^i - \lambda} = \lim_{\lambda\to\infty}\prod_{\mu=1}^{\infty}(1 + \frac{\lambda_\mu - \lambda_\mu^i}{\lambda_\mu^i - \lambda}) = 1. \tag{18}$$

We see from (18) and (17) that

$$\frac{\bar{C}C}{\bar{C}_i C}\prod_{\mu=1}^{\infty}\frac{\lambda_\mu^i}{\lambda_\mu}=1. \tag{19}$$

At last, we can obtain the final expression for γ_{λ_n}. Substituting (19) into (16) we have the formula (14)– the system of M equations ($i=1,...,M$) for determining M components of γ_{λ_n}.

In the one-channel case the formula (14) goes over into the known expression for two spectra:

$$(\lambda_n^2-\lambda_n^1)^{-1}\prod_{\mu=1}^{\infty}{}'\frac{\lambda_\mu^1-\lambda_n^1}{\lambda_\mu^2-\lambda_n^1}(h_2-h_1)\gamma_{\lambda_n}^2=1, \tag{20}$$

where the matrix values become scalars, and we denote, by indeces 1 and 2, two spectra determining scalar norming factor γ_{λ_n}.

The system (14) is not linear one: each row in it contains the quadratic form $\bar{\gamma}_{\lambda_n}(\hat{h}_i-\hat{h})\gamma_{\lambda_n}$. Hence, these equations cannot be solved uniquely in general (including solvability itself). In other words, we have to impose some constraint on choosing the matrices $\hat{h}_i$, i.e. the difference $\hat{h}_i-\hat{h}$. Among other possibilities, we give several realizations which will allow a unique solvability of the system (14).

i) The symmetric matrix $\hat{h}_i-\hat{h}\equiv\hat{\xi}^{(i)}$ has the form of a Jacobi matrix:

$$\hat{\xi}^{(i)}=\begin{pmatrix}
\xi_{11}^{(i)} & \xi_{12}^{(i)} & 0 & 0 & . & . & 0\\
\xi_{12}^{(i)} & 0 & \xi_{23}^{(i)} & 0 & . & . & .\\
0 & \xi_{23}^{(i)} & 0 & \xi_{34}^{(i)} & . & . & .\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & \xi_{M-1M}^{(i)}\\
0 & . & . & . & 0 & \xi_{M-1M}^{(i)} & 0
\end{pmatrix}, \tag{21}$$

where the main diagonal contains only one non-zero element, $\xi_{11}^{(i)}$. Then

$$\bar{\gamma}_{\lambda_n}(\hat{h}_i-\hat{h})\gamma_{\lambda_n}=\xi_{11}^{(i)}\gamma_1(\lambda_n)^2+2\sum_{k\neq 1}^{M}\xi_{k-1k}^{(i)}\gamma_{k-1}(\lambda_n)\gamma_k(\lambda_n). \tag{22}$$

Introducing the variables $\omega_1\equiv\gamma_1(\lambda_n)^2$ and $\omega_k\equiv\gamma_{k-1}(\lambda_n)\gamma_k(\lambda_n),\;\; k=2,...M$ we can rewrite the last expression as follows

$$\bar{\gamma}_{\lambda_n}(\hat{h}_i-\hat{h})\gamma_{\lambda_n}=\xi_{11}^{(i)}\omega_1+2\sum_{k\neq 1}^{M}\xi_{k-1k}^{(i)}\omega_k. \tag{23}$$

Then (14) becomes the system of linear algebraic equations for the variables ω. If $\omega_1=\gamma_1(\lambda_n)^2>0$, then $\gamma_1(\lambda_n)=\pm\omega_1^{1/2}$, $\gamma_2(\lambda_n)=\mp\omega_2/\omega_1^{1/2}$ and so forth. The sign

in front of $\omega_1^{1/2}$ in the expression for $\gamma_1(\lambda_n)$ determines the common sign for γ_{λ_n} and, hence, is inessential: The whole vector-valued wave function is determined to within sign ($\pm$). With the non-zero element $\xi_{ll}^{(i)} \neq 0,\ l \neq 1$ positioned in arbitrary place of the main diagonal, the scheme is analogous.

ii) The matrix $\hat{h}_i - \hat{h} \equiv \hat{\zeta}^{(i)}$ is represented as follows:

$$\hat{\zeta}^{(i)} = \begin{pmatrix} 0 & . & 0 & \zeta_{1l}^{(i)} & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & 0 & \zeta_{l-1l}^{(i)} & 0 & . & . \\ \zeta_{l1}^{(i)} & . & \zeta_{ll-1}^{(i)} & \zeta_{ll}^{(i)} & \zeta_{ll+1}^{(i)} & . & \zeta_{lM}^{(i)} \\ . & . & 0 & \zeta_{l+1l}^{(i)} & 0 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & 0 & \zeta_{Ml}^{(i)} & 0 & . & 0 \end{pmatrix}, \tag{24}$$

i.e., the matrix contains one non-zero row and one non-zero column which cross each other in a place of the entry $\zeta_{ll}^{(i)}$. For the quadratic form we have (using the symmetry of $\hat{h}_i - \hat{h}$)

$$\bar{\gamma}_{\lambda_n}(\hat{h}_i - \hat{h})\gamma_{\lambda_n} = \zeta_{ll}^{(i)}\gamma_l(\lambda_n)^2 + 2\sum_{k\neq l}^{M}\zeta_{lk}^{(i)}\gamma_l(\lambda_n)\gamma_k(\lambda_n). \tag{25}$$

Introducing new variables $\theta_k \equiv \gamma_l(\lambda_n)\gamma_k(\lambda_n),\ \ k \neq l$ and $\theta_l \equiv \gamma_l(\lambda_n)^2$ we can now look upon (14) as a linearized system again:

$$(\lambda_n^i - \lambda_n)^{-1}\prod_{\mu=1}^{\infty}{}'\frac{\lambda_\mu - \lambda_n}{\lambda_\mu^i - \lambda_n}\{\zeta_{ll}^{(i)}\theta_l + 2\sum_{k\neq l}^{M}\zeta_{lk}^{(i)}\theta_k\} = 1. \tag{26}$$

After deriving θ_i, one can obtain $\gamma_i(\lambda_n)$ trivially. Of course, the solvability in this case depends on whether the corresponding determinant for the system (26) is non-zero and $\theta_l > 0$.

In all the cases, the knowledge of the complete set $\{\lambda_n, \gamma_{\lambda_n}\}_{n=1}^{\infty}$ allows a unique restoration of the potential matrix by the standard Gel'fand-Levitan theory (its multichannel generalization).

3 Conclusions

In this paper, the relationship is established between components of the norming vector γ_{λ_n} associated with a certain boundary value problem (with the spectrum $\{\lambda_n\}_{n=1}^{\infty}$) and the spectra (including $\{\lambda_n\}_{n=1}^{\infty}$) of $M+1$ multichannel Sturm-Liouville operators with the same potential matrix $V_{\alpha\beta}(x)$ but different boundary conditions. As a matter of fact, the central result is the formula (14). Though giving no unique solutions in general, it can get linear if we require the matrices $\hat{h}_i$ to be of special type. Hence, the uniqueness of the multichannel inverse eigenvalue problem from $M+1$ spectra is however possible

for a particular class of boundary conditions. The problem of specifying the necessary and sufficient conditions needs a special examination. It is clear that scrutinizing the asymptotic behaviour of the spectra with different boundary conditions will be required. It is closely associated with specifying the class of differentiable functions the $V_{\alpha\beta}(x)$ pertain to. So, the results given present only an intermediate stage in investigations on the subject.

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